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DIFFERENTIABLE INVARIANTS

JOHN N. MATHER*

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§1. INTRODUCTION

G. W. SCHWARZ has proved [10] a fundamental theorem on differentiable invariants, an analogue of Hilbert's classical theorem on polynomial invariants. In this paper, we will strengthen the conclusion of Schwarz's theorem, and at the same time give a simplified proof. However, the main ideas come from Schwarz's paper [10]. His paper is not a prerequisite for the reading of this paper. A key step in our proof is our variant of Borel's lemma (§7). We will simplify Schwarz's proof in other ways, as well.

In a preliminary section (§2), we review some of the classical results related to Hilbert's theorem. The main results of this paper are stated in §3. The chief distinction between our results and Schwarz's results is that we show that the continuous linear mappings arising from Schwarz's theory are *split* surjective, while Schwarz only shows that they are surjective. It appears unlikely that one can obtain split surjectivity from Schwarz's proof.

In §§4,6, we show that the two theorems stated in §3 are equivalent. In §§7–9, we finish the proofs of the main theorems.

Finally, I should mention that a very clear talk by F. Ronga on Schwarz's theorem at Plans-sur-Bex in March 1975 led me to consider the problem of improving Schwarz's result to give split surjectivity.

Sections 2–6 are mainly expository, since most of what is done there is in [10] or earlier papers. Our new ideas are in §§7–9.

In [13], Luna has given another generalization of Schwarz's theorem. In §10, we show that Luna's theorem can be generalized in the same way we have generalized Schwarz's theorem. This generalization was suggested by the referee of this paper. I would like to thank the referee who suggested a number of improvements in the exposition, as well as the result in §10.

§2. POLYNOMIAL AND POWER SERIES INVARIANTS

Throughout this paper, we consider a compact Lie group G acting smoothly on a manifold. ("Smooth" will always mean C^∞.) In this section, we will make the further hypothesis that our action is an orthogonal action on Euclidean space $\mathbf{R}^n$. We will recall Hilbert's theorem in this context. Actually, Hilbert proved his theorems for linear groups, not for compact groups (cf. [5]), but Weyl has given a particularly lucid account of Hilbert's theorems in this context [12]. Moreover, it is this case of Hilbert's theorem that can be generalized to smooth functions.

First, some notation. All functions will be real-valued. If G acts on X, and f is a function on W, then f is said to be *invariant* if $f(gx) = f(x)$ for all $g \in G$ and all $x \in X$. If $\mathcal{F}$ is a set of functions on X, then the subset of invariant members of $\mathcal{F}$ is denoted $\mathcal{F}^G$.

We suppose G acts orthogonally on $\mathbf{R}^n$, in this section. We let $P(\mathbf{R}^n)$ denote the $\mathbf{R}$-algebra of all polynomial functions on $\mathbf{R}^n$. Hilbert's theorem asserts that $P(\mathbf{R}^n)^G$ is finitely generated as an $\mathbf{R}$-algebra (cf. Weyl [12]). A finite generating set will be called a *Hilbert basis* of $P(\mathbf{R}^n)^G$.

Since the homogeneous parts of an invariant polynomial are themselves also invariant, we may choose a Hilbert basis consisting of homogeneous polynomials. Such a Hilbert basis will be said to be *homogeneous*. It will be said to be *minimal* if no proper subset of it is still a Hilbert basis.

We will need three simple consequences of Hilbert's theorem, which are listed in the three

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lemmas below. Let $\sigma_1, \dots, \sigma_k$ be a minimal homogeneous Hilbert basis of $P(\mathbf{R}^n)^G$. Let $\sigma^*: P(\mathbf{R}^k) \to P(\mathbf{R}^n)^G$ be defined by $\sigma^* f = f \circ \sigma$, where $\sigma = (\sigma_1, \dots, \sigma_k)$: $R^n \to R^k$. Hilbert's theorem asserts that σ^* is surjective. Let Π_k denote the ideal in $P(R^k)$ consisting of all f for which $f(0) = 0$. Clearly σ^* induces a surjective linear mapping $\Pi_k/\Pi_k^2 \to \Pi_n^G/(\Pi_n^G)^2$.

LEMMA 1. *This mapping is an isomorphism.*

Proof. We only have to prove that it is injective. If not, we would have a linear combination $\Sigma a_i \sigma_i$ of the σ_i with not all coefficients $= 0$, such that $\Sigma a_i \sigma_i \in (\Pi_n^G)^2$. Suppose $a_i \neq 0$. Then

$$\sigma_i = \sum_{j \neq i} b_j \sigma_j + \tau, \qquad b_j \in \mathbf{R}, \qquad \tau \in (\Pi_n^G)^2.$$

If we think of the right side as a sum of monomials in the σ_j, and drop all monomials of degree different from $\deg \sigma_i$, we obtain an expression of σ_i as a polynomial in σ_j, $j \neq i$, which contradicts the hypothesis that $\sigma_1, \dots, \sigma_k$ is minimal. □

Let d be the maximum degree of any of the σ_i.

LEMMA 2. $\Pi_n^G \cap \Pi_n^{d+1} \subset (\Pi_n^G)^2$.

Proof. Let $u \in \Pi_n^G \cap \Pi_n^{d+1}$ be homogeneous, and write it as a sum of monomials in the σ_i, all of the same degree as u. Since any such monomial must evidently be a product of two or more σ_i, it follows that $u \in (\Pi_n^G)^2$. Since any element of $\Pi_n^G \cap \Pi_n^{d+1}$ is a sum of homogeneous elements, we obtain the desired inclusion. □

We let $F(\mathbf{R}^n)$ denote the $\mathbf{R}$ algebra of formal power series in the coordinates of $\mathbf{R}^n$ with coefficients in $\mathbf{R}$. To topologize $F(\mathbf{R}^n)$, we identify $F(\mathbf{R}^n)$ with $\mathbf{R}^\infty$ by identifying a formal power series $\Sigma c_\alpha x^\alpha$ with the collection $\{c_\alpha\}$ of all its coefficients. Here α denotes a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We give $\mathbf{R}$ the ordinary topology (associated to the metric $|x - y|$) and we give $\mathbf{R}^\infty$ the product topology. We give $F(\mathbf{R}^n)$ the topology which makes the above identification a homeomorphism.

If $\varphi: E \to F$ is a continuous linear mapping of topological vector spaces over $\mathbf{R}$, we say it is *split surjective* if there is a continuous $\mathbf{R}$-linear mapping $\eta: F \to E$ such that $\varphi\eta = \mathrm{id}(F)$.

We let $F(\mathbf{R}^n)^G$ denote the formal power series on $\mathbf{R}^n$ which are invariant under G. We define $\sigma^*: F(\mathbf{R}^k) \to F(\mathbf{R}^n)^G$ by $\sigma^* f = f \circ \sigma$, where the latter means the power series obtained from f by substituting σ_i for the ith coordinate of $\mathbf{R}^k$.

LEMMA 3. *$\sigma^*: F(\mathbf{R}^k) \to F(\mathbf{R}^n)^G$ is split surjective.*

Proof. Let H_d^G denote the vector space of homogeneous invariant polynomials on $\mathbf{R}^n$ of degree d. Let $x_1, \dots, x_k$ be the coordinates of $\mathbf{R}^k$ and assign x_i weight $d_i = \deg \sigma_i$.

Let W_d be the polynomials on $\mathbf{R}^k$ which are weighted homogeneous of degree d. Hilbert's theorem implies $\sigma_d^*: W_d \to H_d^G$ is surjective. Since W_d and H_d^G are finite dimensional vector spaces, it follows that σ_d^* is split surjective. But $\sigma^* = \prod_d \sigma_d^*: \prod_d W_d \to \prod_d H_d^G$, so it follows that σ^* is split surjective. □

§3. THE RESULTS

We will state three theorems, which are the main results of this paper.

First, we consider a compact group G acting orthogonally on $\mathbf{R}^n$, and let $\sigma_1, \dots, \sigma_k$ be a minimal homogeneous Hilbert basis for $P(\mathbf{R}^n)^G$. As before, we let $\sigma = (\sigma_1, \dots, \sigma_k)$: $\mathbf{R}^n \to \mathbf{R}^k$.

For any manifold X, we denote $\mathscr{C}(X)$ the Fréchet space of C^∞ functions on X, provided with the C^∞ topology. By manifold we will always mean a smooth Hausdorff manifold, with a countable basis for its topology, and we will suppose that all components have the same dimension.

THEOREM 1. *The induced mapping $\sigma^*: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(\mathbf{R}^n)^G$ is split surjective.*

Schwarz proved that this mapping is surjective.

Let G be a compact Lie group acting on a smooth manifold X. The orbit space X/G will be provided with the quotient topology. A function $f: X/G \to \mathbf{R}$ will be called *smooth* if $f \circ \pi$ is smooth, where $\pi: X \to X/G$ denotes the projection. We let $\mathscr{C}(X/G)$ denote the $\mathbf{R}$-algebra of all smooth functions on X/G. Clearly $\pi^*: \mathscr{C}(X/G) \to \mathscr{C}(X)^G$ is an isomorphism. We provide $\mathscr{C}(X)^G$ with the C^∞ topology, and topologize $\mathscr{C}(X/G)$ so π^* is a homeomorphism.

If $x \in X/G$, we let $\mathscr{M}_x$ denote the ideal in $\mathscr{C}(X/G)$ consisting of functions vanishing at x. We define the Zariski tangent space to X/G at x as $T(G/X)_x = (\mathscr{M}_x/\mathscr{M}_x^2)^*$. Here V^* means the vector space of *continuous* linear mappings of V into $\mathbf{R}$, and $\mathscr{M}_x/\mathscr{M}_x^2$ is provided with the induced topology.

A mapping $f: X/G \to Y/H$ between orbit spaces will be said to be *smooth* if $u \in \mathscr{C}(Y/H)$ implies $u \circ f \in \mathscr{C}(X/G)$. It is easily seen that if f is smooth, then it is continuous. Moreover, for any $x \in X/G$, we define the derivative $\mathrm{d}f_x: T(X/G)_x \to T(Y/H)_{f(x)}$ as the dual of $f^*: \mathscr{M}_{f(x)}/\mathscr{M}^2_{f(x)} \to \mathscr{M}_x/\mathscr{M}_x^2$.

These definitions apply in the particular case when H is reduced to a point, so $Y/H = Y$.

LEMMA. *Let G be a compact Lie group, acting orthogonally on $\mathbf{R}^n$. Let $\sigma_1, \ldots, \sigma_k$ be a minimal homogeneous Hilbert basis of $P(\mathbf{R}^n)^G$. Let $\sigma = (\sigma_1, \ldots, \sigma_k)$ and let $\bar{\sigma}: \mathbf{R}^n/G \to \mathbf{R}^k$ denote the induced mapping. Then $\mathrm{d}\bar{\sigma}_0: T(\mathbf{R}^n/G)_0 \to T(\mathbf{R}^k)_0$ is an isomorphism.*

Proof. Let $\mathscr{M}_l$ denote the ideal in $\mathscr{C}(\mathbf{R}^l)$ consisting of all functions which vanish at 0, and let $\hat{\mathscr{M}}_l$ denote the maximal ideal in $F(\mathbf{R}^l)$. We have a commutative diagram

$$\begin{array}{ccc}
\hat{\mathscr{M}}_k/\hat{\mathscr{M}}_k^2 & \longrightarrow & \hat{\mathscr{M}}_n^G/(\hat{\mathscr{M}}_n^G)^2 \\
\uparrow & & \uparrow \\
\mathscr{M}_k/\mathscr{M}_k^2 & \longrightarrow & \mathscr{M}_n^G/(\mathscr{M}_n^G)^2 \\
\uparrow & & \uparrow \\
\Pi_k/\Pi_k^2 & \longrightarrow & \Pi_n^G/(\Pi_n^G)^2
\end{array}$$

where the horizontal arrows are induced by σ^* and the vertical arrows are induced by the inclusion of the polynomials in the smooth functions, and the Taylor homomorphism, respectively. From Lemma 1 in §2, it follows that the bottom horizontal arrow is an isomorphism. Using Lemmas 1 and 2 in §2, one easily checks that the top horizontal arrow is an isomorphism. The left vertical arrows are isomorphisms. The composition of the right vertical arrows is an isomorphism, and the kernel of the vertical arrow in the upper right corner is the closure of 0. Thus, we see that the induced mapping

$$\mathscr{M}_k/\mathscr{M}_k^2 \to \frac{\mathscr{M}_n^G/(\mathscr{M}_n^G)^2}{\bar{0}}$$

is an isomorphism. The lemma follows immediately. □

Note. From Schwarz's theorem, it follows that $\bar{0} = 0$, but we are unable to prove this without using Schwarz's theorem. This is why we defined the dual as *continuous* linear functions.

Definition. $f: X/G \to Y$ is an *embedding* if it is smooth, proper, one–one, and $\mathrm{d}f_x$ is injective for any $x \in X/G$.

Now we state the other two main results which we will prove.

THEOREM 2. *If $f: X/G \to Y$ is an embedding, then f^* is split surjective.*

THEOREM 3. *If $f: X/G \to Y$ is proper and smooth, and $f^*\mathscr{C}(Y)$ is dense in $\mathscr{C}(X/G)$, then f is an embedding.*

Theorems 1 and 2 will be proved in later sections. Here, we prove Theorem 3.

Proof of Theorem 3. It is easy to see that $\mathscr{C}(X/G)$ separates points in X/G. For, let x, y be distinct points in X/G, and let $\pi: X \to X/G$ denote the projection. Since $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are disjoint closed subsets of the manifold X, there exists a smooth function u on X such that $u|\pi^{-1}(x) = 0$ and $u|\pi^{-1}(y) = 1$. By averaging over G, we may assume u is invariant. Thus, we get a function $\bar{u} \in \mathscr{C}(X/G)$ such that $\bar{u}(x) = 0$ and $\bar{u}(y) = 1$.

Since $\mathscr{C}(X/G)$ separates points, and $f^*\mathscr{C}(Y)$ is dense in $\mathscr{C}(X/G)$, it follows that $f^*\mathscr{C}(Y)$ separates points. It follows that f is one–one.

Let $x \in X/G$, and set $y = f(x)$. Clearly $f^*(\mathscr{M}_y/\mathscr{M}_y^2)$ is dense in $\mathscr{M}_x/\mathscr{M}_x^2$. It follows that the induced mapping

$$\frac{\mathcal{M}_y}{\mathcal{M}_y^2} \to \frac{\mathcal{M}_x/\mathcal{M}_x^2}{\bar{0}}$$

is surjective, since the vector space on the right is finite dimensional and Hausdorff. Passing to the duals, we obtain that $df_x: TX_x \to TY_{f(x)}$ is injective. □

§4. THEOREMS 2 AND 3 IMPLY THEOREM 1

First, σ is proper, since $x_1^2 + \cdots + x_n^2$ is invariant and proper, and $x_1^2 + \cdots + x_n^2$ is a polynomial in $\sigma_1, \ldots, \sigma_k$.

By Theorems 2 and 3, it is then enough to show that $\sigma^*\mathscr{C}(\mathbf{R}^k)$ is dense in $\mathscr{C}(\mathbf{R}^n)^G$. The set $P(\mathbf{R}^n)$ is dense in $\mathscr{C}(\mathbf{R}^n)$, so by averaging over G, we see that $P(\mathbf{R}^n)^G$ is dense in $\mathscr{C}(\mathbf{R}^n)^G$. Since $\sigma^*: P(\mathbf{R}^k) \to P(\mathbf{R}^n)^G$ is surjective, we deduce that $\sigma^*P(\mathbf{R}^k)$ is dense in $\mathscr{C}(\mathbf{R}^n)^G$. Hence $\sigma^*\mathscr{C}(\mathbf{R}^k)$ is dense in $\mathscr{C}(\mathbf{R}^n)^G$. □

Our argument also shows that $\sigma: \mathbf{R}^n/G \to \mathbf{R}^k$ is proper embedding.

§5. EXISTENCE OF MANY EMBEDDINGS

As pointed out by Schwarz[10], most G-spaces have the property that their orbit spaces can be embedded in Euclidean space. Let G act on X. If $x \in X$, let G_x denote the isotropy subgroups of x (i.e. $\{g \in G: gx = x\}$), O_x the orbit through x, and E_x the normal space to O_x in X at x. The differential defines a representation ρ_x of G_x in E_x. Two points x and y in X are said to be of the same *slice type* if there is an inner automorphism I of G such that $G_y = I(G_x)$ and the representations $\rho_y I$ and ρ_x are equivalent.

By a theorem of Mann[2, 7], any action of a compact group on X will have only a finite number of slice types, if the integral homology of X is finitely generated. On the assumption that G acts on X with only finitely many slice types, we will show that X/G embeds in $\mathbf{R}^k$ for sufficiently large k.

By a theorem of Mostow and Palais[8, 9, 2], there is an orthogonal action of G on $\mathbf{R}^N$, for suitable N, and an equivariant proper embedding e of X in $\mathbf{R}^N$. It is easily seen that $e^*: \mathscr{C}(\mathbf{R}^N)^G \to \mathscr{C}(X)^G$ is surjective: to extend an invariant function on X, first extend a function which is not necessarily invariant, and then average. Let $\sigma_1, \ldots, \sigma_k$ be a minimal Hilbert basis for $P(\mathbf{R}^N)^G$. Since $\sigma^*: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(\mathbf{R}^N)^G$ has dense image, it follows easily that $(\sigma e)^*: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(X)^G$ has dense image. It then follows from Theorem 3 that the mapping $\overline{\sigma e}: X/G \to \mathbf{R}^k$ induced by σe is an embedding.

§6. THEOREM 1 IMPLIES THEOREM 2

We actually show more. Let R_n be the assertion that Theorem 1 holds for all orthogonal actions on $\mathbf{R}^p$, where $p \le n$. Let T_n be the assertion that Theorem 2 holds for all smooth actions on manifolds of dimension $\le n$, and all embeddings of the associated quotient spaces. In this section, we will show $R_n \Leftrightarrow T_n$.

Throughout this section, we consider an action of a compact Lie group G on a manifold X. We suppose $\dim X \le n$. We will also consider a smooth embedding $f: X/G \to Y$. Assuming R_n, we will show that $f^*: \mathscr{C}(Y) \to \mathscr{C}(X/G)$ is split surjective.

A standard partitions of unity argument now reduces the problem of showing that f^* is split surjective to a local problem.

Definition. Let $x \in X/G$. We will say f^* is *locally split surjective at* x if there is an open neighborhood U of x in X/G and a continuous linear mapping $l: \mathscr{C}(X/G) \to \mathscr{C}(Y)$ such that $f^*l(u)|U = u|U$ for any $u \in \mathscr{C}(X/G)$. We will say that f^* is *locally split surjective* if it is locally split surjective at each point of X/G.

LEMMA 1. *If f^* is locally split surjective, then it is split surjective.*

Proof. Since $f(X/G)$ is closed in Y and Y is paracompact, we can choose a locally finite open cover $\{W_\alpha\}$ of Y such that for each α, there is a continuous linear mapping $l_\alpha: \mathscr{C}(X/G) \to \mathscr{C}(Y)$ such that $f^*l_\alpha(u)|f^{-1}(W_\alpha) = u|f^{-1}(W_\alpha)$, for any $u \in \mathscr{C}(X/G)$. Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to $\{W_\alpha\}$. We define $l: \mathscr{C}(X/G) \to \mathscr{C}(Y)$ by $l(u) = \sum_\alpha \rho_\alpha l_\alpha(u)$. It is easily verified that l is continuous and $f^* l = \mathrm{id}$. □

Now we recall the *slice theorem*. We may construct an invariant metric on X by taking any Riemannian metric and averaging over G. Let $x \in X$, and let O_x denote the orbit through x. Let E_x denote the vector space of tangent vectors at x, perpendicular to O_x. Let G_x denote the isotropy subgroup of x. For $\epsilon > 0$, let E_x^ϵ denote the vectors in E_x of norm $<\epsilon$. The slice theorem asserts that if $\epsilon > 0$ is sufficiently small, then the mapping $(g, v) \to g \cdot \exp(v)$ gives a G-equivariant diffeomorphism of $G \times_{G_x} E_x^\epsilon$ onto an open invariant neighborhood U_x of O_x in X.

It follows that there is a diffeomorphism $\varphi: U_x/G \to E_x^\epsilon/G_x$ (i.e. a smooth mapping with a smooth inverse, where smooth is taken in the sense defined in §3).

Since E_x is a Euclidean space and G_x acts orthogonally on it, we may choose a minimal Hilbert basis $\sigma_1, \dots, \sigma_k$ of $P(E_x)^{G_x}$. By our assumption that R_n holds, $\sigma^*: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(E_x)^{G_x}$ is split surjective. Hence $\sigma^*: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(E_x^\epsilon)^{G_x}$ is locally split surjective at 0.

Let $y = f(\bar{x})$, where $\bar{x}$ is the image of x under the projection $\pi: X \to X/G$. Let $y_1, \dots, y_p$ be a smooth local system of coordinates on Y, defined in a neighborhood of y. Since $\sigma^*: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(E_x^\epsilon)^{G_x}$ is locally surjective at 0, there exist smooth functions $u_1, \dots, u_p$, defined in a neighborhood of 0 in $\mathbf{R}^k$, such that $\sigma^* u_i = y_i \circ f \circ \varphi^{-1}$ in a sufficiently small neighborhood of 0 in E_x/G_x^ϵ.

Let $u: \mathbf{R}^k \to Y$ be defined by $y_i \circ u = u_i$; then u is defined in a neighborhood of 0. We have a commutative diagram.

$$\begin{array}{ccc} U_x/G & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle u} \\ E_x^\epsilon/G_x & \xrightarrow{\bar{\sigma}} & \mathbf{R}^k \end{array}$$

Here u is only defined in a neighborhood of 0.

We have that $d\varphi_{\bar{x}}$ is an isomorphism, since φ is a diffeomorphism. Furthermore $d\bar{\sigma}_0$ is an isomorphism, by the lemma in §3. Moreover, $df_{\bar{x}}$ is injective, since f is an embedding. Since $df_{\bar{x}} = du_0 \cdot d\bar{\sigma}_0 \cdot d\varphi_{\bar{x}}$, it follows that du_0 is injective. Hence $u^*: \mathscr{C}(Y) \to \mathscr{C}(\mathbf{R}^k)$ is locally split surjective at 0.

Now $f^* = \varphi^* \bar{\sigma}^* u^*$ is a composition of locally split surjective homomorphisms. Hence f^* is locally split surjective at $\bar{x}$. Since $\bar{x}$ is an arbitrary point of X/G, this shows f^* is split surjective.

§7. A VARIANT OF BOREL'S LEMMA

In this section, we prove a lemma which is the chief novelty of this paper. Once we have this lemma, our strong form of Schwarz's theorem follows quite easily. We will use this lemma in §9.

Let $T: \mathscr{C}(\mathbf{R}^n) \to F(\mathbf{R}^n)$ be the Taylor homomorphism, i.e., Tf = the Taylor series expansion of f at 0. An elementary lemma of E. Borel asserts that T is surjective. Unfortunately, T is not split surjective (cf. [4a], IV).

Therefore, the following lifting problem is interesting. Let $F: \mathscr{C}(\mathbf{R}^k) \to F(\mathbf{R}^n)$ be a continuous linear mapping. We wish to solve:

Lifting Problem. Find a continuous linear $\tilde{F}: \mathscr{C}(\mathbf{R}^k) \to \mathscr{C}(\mathbf{R}^n)$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{C}(\mathbf{R}^k) \arrow[d, dashed, "\tilde{F}"'] \arrow[dr, "F"] & \\ \mathscr{C}(\mathbf{R}^n) \arrow[r, "T"] & F(\mathbf{R}^n). \end{tikzcd} \qquad \text{(diag. 3)}$$

Let $x \in \mathbf{R}^k$. We will say F is *null at* x if there exists a neighborhood U of x in $\mathbf{R}^k$ such that if $f \in \mathscr{C}(\mathbf{R}^k)$ and $\operatorname{supp} f \subset U$, then $F(f) = 0$. Clearly, the set of points at which F is null is open. By the *support* of F we mean the complement of the set of points where F is null. We denote this set by supp F. Clearly supp F is closed. This definition of support generalizes the standard definition of the support of a distribution.

Now we can state our lemma.

LEMMA. *If F has compact support, then there exists a continuous linear $\tilde{F}$ which makes diagram 3 commutative.*

Proof. Let p be a smooth function on $\mathbf{R}^k$ with compact support and values in $[0, 1]$, such that p is identically 1 in a neighborhood of supp F. Let ρ be a smooth function on $\mathbf{R}^n$ with support in the unit ball, invariant under the action of the orthogonal group on $\mathbf{R}^n$, and identically 1 in a neighborhood of 0. For any $\lambda > 0$, let ρ_λ be the function on $\mathbf{R}^n$ defined by $\rho_\lambda(x) = \rho(\lambda x)$.

Let K be a large positive number, such that p has support in the interior of the cube of side K centered at 0 in $\mathbf{R}^k$. For any $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{Z}^k$, let $e_\alpha(x) = e^{2\pi i\alpha \cdot x/K}$, if $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. Then $e_\alpha \in \mathscr{E}(\mathbf{R}^k)$. Let $\epsilon_\alpha = F(e_\alpha)$. Let $\epsilon_{\alpha,r}$ be the homogeneous part of order r of ϵ_α. Then $\epsilon_\alpha = \epsilon_{\alpha,0} + \cdots + \epsilon_{\alpha,r} + \cdots$.

We will choose later, for each $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{Z}^k$ and each $r \geq 0$, a positive number $\lambda(\alpha, r)$. We let D_K denote the cube of side K centered at 0. If $u \in \mathscr{E}(\mathbf{R}^n)$, we expand $pu|D_K$ in a Fourier series

$$pu|D_K = \sum_\alpha c_\alpha e_\alpha, \qquad c_\alpha \in C.$$

Then we define

$$\tilde{F}(u) = \sum_{\alpha,r} c_\alpha \rho_{\lambda(\alpha,r)} \epsilon_{\alpha,r}, \tag{2}$$

where we think of $\epsilon_{\alpha,r}$ as a function on $\mathbf{R}^n$, which we may, since it is a homogeneous polynomial of degree r.

Of course, if we make a bad choice of the $\lambda(\alpha, r)$, the sum on the right-hand side of (2) may not converge. We will show that by choosing the $\lambda(\alpha, r)$ suitably, we may arrange that the sum on the right-hand side converges with respect to the C^∞ topology on $\mathscr{E}(\mathbf{R}^n)$, and that the resulting mapping $\tilde{F}$: $\mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n)$ is continuous. Obviously $\tilde{F}$ is linear if it can be defined in this way, and $T\tilde{F} = F$.

We will also show that we may choose the $\lambda(\alpha, r)$ so that $\lambda(\alpha, r) = \lambda(\alpha', r)$ if $|\alpha_i| = |\alpha'_i|$, $i = 1, \ldots, k$. Then, if u is real-valued, so is $\tilde{F}(u)$. Clearly, if we can choose the $\lambda(\alpha, r)$ so that all these conditions are satisfied, then we will have proved the lemma.

First, we estimate the size of the $\epsilon_{\alpha,r}$. We let $P_r(\mathbf{R}^n)$ denote the vector space of all homogeneous polynomials on $\mathbf{R}^n$ of degree r. We let n denote any norm on $P_r(\mathbf{R}^n)$: since $P_r(\mathbf{R}^n)$ is finite-dimensional, any two norms on it are equivalent. We let F_r: $\mathscr{E}(\mathbf{R}^k) \to P_r(\mathbf{R}^n)$ be defined by $F_r(u) = F(u)_r$, where the latter denotes the homogeneous part of order r of $F(u)$. Since F is linear, continuous, and has compact support, it follows that F_r has the same properties. Thus, F_r is a $P_r(\mathbf{R}^n)$-valued distribution on $\mathbf{R}^k$ with compact support. It follows from a standard estimate in the theory of distributions (cf. [6], (1.5.4)), that

$$n(F_r(u)) \leq C_r \sum_{0 \leq \beta \leq s(r)} \sup_{D_K} \|D^\beta u\|, \qquad u \in \mathscr{E}(\mathbf{R}^k),$$

where C_r and $s(r)$ are constants, and D^β denotes the total derivative of order β, defined as for instance in [3]. From this, we obtain immediately

$$n(\epsilon_{\alpha,r}) = n(F(e_\alpha)_r) \leq C'_r(1 + |\alpha|)^{s(r)}, \tag{3}$$

where C'_r is a constant.

Now we estimate the lth total derivative of $\tilde{F}(u)$. From (2), and Leibniz's formula, we obtain

$$\tilde{F}(u)^{(l)} = \sum_{\alpha,r} c_\alpha \sum_{0 \leq m \leq l} \binom{l}{m} \rho^{(m)}_{\lambda(\alpha,r)} \epsilon^{(l-m)}_{\alpha,r}.$$

Clearly $\rho^{(m)}_{\lambda(\alpha,r)}$ has support in the ball of radius $\lambda(\alpha, r)^{-1}$ and $\sup \|\rho^{(m)}_{\lambda(\alpha,r)}\| \leq C''_m \lambda(\alpha, r)^m$, where $C''_m = \sup \|\rho^{(m)}\|$. Letting A denote the ball of radius $\lambda(\alpha, r)^{-1}$, we find, in view of (3), and the fact that $\epsilon_{\alpha,r}$ is a homogeneous polynomial of degree r, that

$$\sup_A \|\epsilon^{(l-m)}_{\alpha,r}\| \leq C'''_r(1 + |\alpha|)^{s(r)} \lambda(\alpha, r)^{l-m-r},$$

where C'''_r is a constant. Thus,

$$\sup \|\tilde{F}(u)^{(l)}\| \le \sum_{\alpha, r} c_\alpha \sum_{0 \le m \le l} \binom{l}{m} C''_m C'''_r (1+|\alpha|)^{s(r)} \lambda(\alpha, r)^{l-r}$$
$$\le \sum_{\alpha, r} c_\alpha C^{iv}_{rl} (1+|\alpha|)^{s(r)} \lambda(\alpha, r)^{l-r}, \tag{4}$$

where

$$C^{iv}_{rl} = \sum_{0 \le m \le l} \binom{l}{m} C''_m C'''_r.$$

Now we choose the $\lambda(\alpha, r)$. For each fixed r, we will choose $\lambda(\alpha, r)$ to be so rapidly increasing with $|\alpha| = |\alpha_1| + \cdots + |\alpha_k|$ that

$$\sum_\alpha C^{iv}_{rl}(1+|\alpha|)^{s(r)} \lambda(\alpha, r)^{l-r} \le 2^{l-r}, \qquad \text{if } l < r, \tag{5}$$

and at the same time, choose $\lambda(\alpha, r)$ so that for fixed r it has at most polynomial growth in $|\alpha|$. This may be achieved, for example, by choosing

$$\lambda(\alpha, r) = C^v_r (1+|\alpha|)^{s(r)+k+1}, \tag{6}$$

where

$$C^v_r = \sup_{l<r} 2\left(C^{iv}_{rl} \sum_\alpha (1+|\alpha|)^{-k-1}\right)^{1/(r-l)}.$$

Since α varies over $\mathbf{Z}^k$, the sum $\Sigma(1+|\alpha|)^{-\rho}$ coverges for any $\rho > k$. Since there are only a finite number of $l < r$, it is then clear that $C^v_r < \infty$. Clearly, $\lambda(\alpha, r)$ satisfies (5) and has at most polynomial growth in $|\alpha|$. Also, it is clear that $|\alpha_i| = |\alpha'_i|$, $i = 1, \ldots, k$ implies $\lambda(\alpha, r) = \lambda(\alpha', r)$.

Now we show that (2) converges in the C^∞ topology. Since the c_α are the Fourier coefficients of $pu|D_k$, and pu is C^∞ and vanishes in a neighborhood of ∂D_K, we have for any $\mu > 0$,

$$(1+|\alpha|)^\mu c_\alpha \to 0, \qquad \text{as } |\alpha| \to \infty. \tag{7}$$

From (4), (5), and (6), we obtain

$$\sup \|\tilde{F}(u)^{(l)}\| \le \sum_{0 \le r \le l} C^{vi}_{rl} \sum_\alpha c_\alpha (1+|\alpha|)^{s'(r,l)} + \sum_{r>l} 2^{l-r} \sum_\alpha c_\alpha, \tag{8}$$

where

$$C^{vi}_{rl} = C^{iv}_{rl}(C^v_r)^{l-r}, \qquad s'(r, l) = s(r) + (l-r)(s(r)+k+1).$$

From (7), it clearly follows that (8) converges. Moreover, if we define, for $u \in \mathscr{C}(\mathbf{R}^k)$, $pu|D_K = \Sigma c_\alpha e_\alpha$,

$$\|u\|_\mu = \sum_\alpha |c_\alpha|(1+|\alpha|)^\mu$$

the $\|\ \|_\mu$ is a continuous semi-norm on $\mathscr{C}(\boldsymbol{R}^k)$. From (8), we obtain

$$\sup \|\tilde{F}(u)^{(l)}\| \le \sum_{0 \le r \le l} C^{vi}_{rl} \|u\|_{s'(r,l)} + \|u\|_0.$$

Therefore $\tilde{F}$ is continuous with respect to the C^∞ topologies on $\mathscr{C}(\mathbf{R}^k)$ and $\mathscr{C}(\mathbf{R}^n)$. This completes the proof of the lemma. □

§8. INVARIANTS WHICH VANISH TO INFINITE ORDER AT 0

Throughout this section, we suppose that T_{n-1} holds. We consider an orthogonal action of a compact Lie group G on $\mathbf{R}^n$, and let $\sigma_1, \ldots, \sigma_k$ be a minimal homogeneous Hilbert basis of $P(\boldsymbol{R}^n)^G$. We let $\mathscr{C}(\boldsymbol{R}^k)_0$ denote the set of C^∞ functions on $\mathbf{R}^k$ which vanish to infinite order at 0. In this section, we will show that $\sigma^*: \mathscr{C}(\mathbf{R}^k)_0 \to \mathscr{C}(\mathbf{R}^n)_0^G$ is split surjective.

We let S^{k-1} denote the unit sphere in $\mathbf{R}^k$. We let $\Sigma^{n-1} = \sigma^{-1}(S^{k-1})$. Each ray emanating from the origin in $\mathbf{R}^n$ meets Σ^{n-1} in exactly one point, transversally. For, consider $x \in \mathbf{R}^n$, $x \neq 0$, and let $\varphi(t) = |\sigma(tx)|^2$ for $t \ge 0$. Then Σ^{n-1} meets the ray through x in exactly those points where $\varphi(t) = 1$. But, $\varphi(t) = t_1^{2d_1}\sigma_1^2(x) + \cdots + t_k^{2d_k}\sigma_k^2(x)$ and not all $\sigma_i(x)$ are 0, since $x_1^2 + \cdots + x_n^2$ is an invariant and therefore can be expressed as a polynomial in the σ_i. Therefore $\varphi(t) = 1$ has exactly one positive solution, and $\varphi'(t) \neq 0$ there, which proves that the ray through x meets Σ^{n-1} in exactly one point, transversally.

Hence Σ^{n-1} is a compact analytic manifold.

We define $r: S^{k-1} \times \mathbf{R} \to \mathbf{R}^k$ by $r(x_1, \ldots, x_k, t) = (t^{d_1}x_1, \ldots, t^{d_k}x_k)$, where $(x_1, \ldots, x_k) \in S^{k-1} \subset \mathbf{R}^k$ and $d_i = \deg \sigma_i$. We define $\rho: \Sigma^{n-1} \times \mathbf{R} \to \mathbf{R}^n$ by $\rho(x, t) = tx$ where $x \in \Sigma^{n-1} \subset \mathbf{R}^n$ and $t \in \mathbf{R}$. We let $\mathbf{R}_+$ be the non-negative real numbers. We let

$$r_+: S^{k-1} \times \mathbf{R}_+ \to \mathbf{R}^k, \qquad \rho_+: \Sigma^{n-1} \times \mathbf{R}_+ \to \mathbf{R}^n$$

denote the restrictions of r and ρ.

We consider Σ^{n-1} as a G-space with respect to the restriction of the given action on $\mathbf{R}^n$. We consider $\mathbf{R}_+$ as a G-space with the trivial action. Then $\Sigma^{n-1} \times \mathbf{R}_+$ is a G-space, and ρ_4 is equivariant.

Moreover, the following diagram commutes:

$$\begin{tikzcd}
S^{k-1} \times \mathbf{R}_+ \arrow[d, "r_+"] & \Sigma^{n-1} \times \mathbf{R}_+ \arrow[l, "\sigma \times \mathrm{id}"'] \arrow[d, "\rho_+"] \\
\mathbf{R}^k & \mathbf{R}^n. \arrow[l, "\sigma"']
\end{tikzcd}$$

If X is any manifold, and K is any subset of X, let $\mathscr{E}(X)_K$ denote the C^∞ functions on X which vanish to infinite order on K.

LEMMA. (1) $r_+^* \mathscr{E}(\mathbf{R}^k)_0 = \mathscr{E}(S^{k-1} \times \mathbf{R}_+)_{S^{k-1} \times 0}$,
(2) $\rho_+^* \mathscr{E}(\mathbf{R}^n)_0^G = \mathscr{E}(\Sigma^{n-1} \times \mathbf{R}_+)^G_{\Sigma^{n-1} \times 0}$.

Proof. In both cases the inclusion $\subset$ is obvious. Moreover, it is clear that the left side is dense in the right, since it contains those functions in the right which vanish in a neighborhood of $S^{k-1} \times 0$, resp. $\Sigma^{n-1} \times 0$. However, by applying Glaeser's theorem [4], we see that the left side is closed in the right side. (Strictly speaking, we should apply Glaeser's theorem to the mappings r and ρ, and then we see that $r^* \mathscr{E}(\mathbf{R}^k)$ and $\rho^* \mathscr{E}(\mathbf{R}^n)$ are closed. But, what we need then follows quickly.) □

By the lemma, it is enough to prove that the top arrow in the following commutative diagram is split surjective:

$$\begin{tikzcd}
\mathscr{E}(S^{k-1} \times \mathbf{R}_+)_{S^{k-1} \times 0} \arrow[r, "(\sigma \times \mathrm{id})^*"] & \mathscr{E}(\Sigma^{n-1} \times \mathbf{R}_+)^G_{\Sigma^{n-1} \times 0} \\
\mathscr{E}(\mathbf{R}^k)_0 \arrow[r, "\sigma^*"] \arrow[u, "r_+^*", "\wr"'] & \mathscr{E}(\mathbf{R}^n)_0^G. \arrow[u, "\rho_+^*"', "\wr"]
\end{tikzcd}$$

For, it is an easy consequence of the lemma that the vertical arrows are topological isomorphisms.

Now if U is an open set in $\mathbf{R}^n$ and F is a Fréchet space, we let $\mathscr{E}(U, F)$ denote the C^∞ functions on U with values in F, provided with the C^∞ topology [11]. We have a commutative diagram

$$\begin{tikzcd}
\mathscr{E}(S^{k-1} \times \mathbf{R}_+)_{S^{k-1} \times 0} \arrow[r, "(\sigma \times \mathrm{id})^*"] & \mathscr{E}(\Sigma^{n-1} \times \mathbf{R}_+)^G_{\Sigma^{n-1} \times 0} \\
\mathscr{E}(\mathbf{R}_+, \mathscr{E}(S^{k-1}))_0 \arrow[r] \arrow[u, "\wr"'] & \mathscr{E}(\mathbf{R}_+, \mathscr{E}(\Sigma^{n-1})^G)_0, \arrow[u, "\wr"']
\end{tikzcd}$$

where the bottom arrow is induced from the mapping $\mathscr{E}(S^{k-1}) \xrightarrow{\sigma^*} \mathscr{E}(\Sigma^{n-1})^G$. But σ^* has dense image by the argument used in §4, so it is split surjective by the hypothesis that T_{n-1} holds. Therefore, the bottom arrow in the above diagram is split surjective, and we deduce that $(\sigma \times \mathrm{id})^*$ is split surjective. But, we have already seen that this is enough. □

§9. END OF PROOF

We will show $T_{n-1} \Leftrightarrow R_n$. We have already shown $R_n \Leftrightarrow T_n$ (§6). Clearly, T_0 holds, so this will be enough.

We consider an orthogonal action of G on $\mathbf{R}^n$ and let $\sigma_1, \ldots, \sigma_k$ be a minimal homogeneous Hilbert basis on $P(\mathbf{R}^n)^G$. We consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}(\mathbf{R}^n)_0{}^G \arrow[r] & \mathscr{E}(\mathbf{R}^n)^G \arrow[r, "T"] & F(\mathbf{R}^n)^G \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{E}(\mathbf{R}^k)_0 \arrow[r] \arrow[u] & \mathscr{E}(\mathbf{R}^k) \arrow[r, "T"] \arrow[u, "\sigma^*"] & F(\mathbf{R}^k) \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

In §2, Lemma 3, we have shown that the right vertical arrow is split surjective. Consider the composition

$$\mathscr{E}(\mathbf{R}^n) \xrightarrow{A} \mathscr{E}(\mathbf{R}^n)^G \xrightarrow{T} F(\mathbf{R}^n)^G \xrightarrow{S} F(\mathbf{R}^k),$$

where A is defined by averaging over G, and S splits $F(\mathbf{R}^k) \to F(\mathbf{R}^n)^G$. By the lemma in §7, we may lift this mapping to a continuous linear mapping η_0: $\mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(\mathbf{R}^k)$. Let $\eta_1 = \eta_0 | \mathscr{E}(\mathbf{R}^n)^G$. Then we have

$$T\sigma^*\eta_1 = T: \mathscr{E}(R^n)^G \to F(R^n)^G.$$

Since the inclusion mapping of $\mathscr{E}(\mathbf{R}^n)_0{}^G$ into $\mathscr{E}(\mathbf{R}^n)^G$ is a homeomorphism onto its image, it therefore follows that $\sigma^*\eta_1 - \mathrm{id}(\mathscr{E}(\mathbf{R}^n)^G)$ may be regarded as a continuous linear mapping into $\mathscr{E}(\mathbf{R}^n)_0{}^G$, i.e.,

$$\sigma^*\eta_1 - \mathrm{id}(\mathscr{E}(\mathbf{R}^n)^G): \mathscr{E}(\mathbf{R}^n)^G \to \mathscr{E}(\mathbf{R}^n)_0{}^G.$$

From the split surjectivity of the left arrow (§8), we can therefore deduce that there exists a continuous linear mapping η_2: $\mathscr{E}(\mathbf{R}^n)^G \to \mathscr{E}(\mathbf{R}^k)_0 \subset \mathscr{E}(\mathbf{R}^k)$ such that $\sigma^*\eta_1 - \mathrm{id}(\mathscr{E}(\mathbf{R}^n)^G) = \sigma^*\eta_2$. Setting $\eta = \eta_1 - \eta_2$: $\mathscr{E}(\mathbf{R}^n)^G \to \mathscr{E}(\mathbf{R}^k)$, we see that $\sigma^*\eta = \mathrm{id}$, so σ^* is split surjective. □

§10 GENERALIZATION OF LUNA'S THEOREM

It was pointed out by the referee of this paper that Luna generalized [13] Schwarz's theorem in another direction, and that the techniques in the preceding sections permits a generalization of one of Luna's theorems, analogous to the generalization we gave above of Schwarz's theorem. In this section we will state and prove this generalization of Luna's theorem. The proof depends on the results in Luna's paper and the results in the preceding sections.

Following Luna, we consider a group Γ acting linearly on $\mathbf{R}^n$ and we suppose that the action is completely reducible. Hilbert's theorem states that $P(\mathbf{R}^n)^\Gamma$ is finitely generated as an $\mathbf{R}$-algebra. Let $\sigma_1, \ldots, \sigma_k$ be a generating set of $P(\mathbf{R}^n)^\Gamma$ as an $\mathbf{R}$-algebra. Let $\sigma = (\sigma_1, \ldots, \sigma_k)$: $\mathbf{R}^n \to \mathbf{R}^k$. Following Luna, we let $\mathscr{E}(\mathbf{R}^n; \sigma)$ denote the smooth functions on $\mathbf{R}^n$ which are constant on the fibers of σ. Our generalization of Luna's theorem may be stated as follows.

THEOREM 4. *σ^*: $\mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n; \sigma)$ is split surjective.*
Luna proved that σ^* is surjective.

Proof. Whether σ^* is split surjective clearly doesn't depend on the choice of Hilbert basis, so we may suppose that $\sigma = (\sigma_1, \ldots, \sigma_k)$ is a minimal homogeneous Hilbert basis of $P(\mathbf{R}^n)^\Gamma$.

We will prove the theorem by induction on n. For $n = 0$, it is obvious. Thus, we may suppose that the theorem holds for any completely reducible action of Γ on $\mathbf{R}^m$, $m < n$.

If $(\mathbf{R}^n)^\Gamma \neq 0$, we choose an invariant complement W of $(\mathbf{R}^n)^\Gamma$ in $\mathbf{R}^n$. By means of a linear change of coordinates, we may suppose $W = \mathbf{R}^m \times 0$ and $(\mathbf{R}^n)^\Gamma = 0 \times \mathbf{R}^{n-m}$. Let $x_1, \ldots, x_{n-m}$ be coordinates for $0 \times \mathbf{R}^{n-m}$. Let $\sigma_1, \ldots, \sigma_l$ $(l = k - m + n)$ be a minimal homogeneous Hilbert basis for $P(\mathbf{R}^m \times 0)^\Gamma$. Then $\sigma_1, \ldots, \sigma_l, x_1, \ldots, x_{n-m}$ is a minimal Hilbert basis for $P(\mathbf{R}^n)^\Gamma$ and

$$\sigma = \bar{\sigma} \times \mathrm{id}: \mathbf{R}^m \times \mathbf{R}^{n-m} \to \mathbf{R}^l \times \mathbf{R}^{n-m} = \mathbf{R}^k,$$

where $\bar{\sigma} = (\sigma_1, \ldots, \sigma_l)$: $\mathbf{R}^m \to \mathbf{R}^l$. By induction hypothesis $\bar{\sigma}$: $\mathscr{E}(\mathbf{R}^l) \to \mathscr{E}(\mathbf{R}^m; \bar{\sigma})$ is split surjective. Hence the top arrow in the diagram below is split surjective:

$$\begin{array}{ccc} \mathscr{C}(\mathbf{R}^{n-m}, \mathscr{C}(\mathbf{R}^l)) & \xrightarrow{\sigma^{*}\,\bar{}} & \mathscr{C}(\mathbf{R}^{n-m}, \mathscr{C}(\mathbf{R}^m; \bar{\sigma})) \\ \| & & \| \\ \mathscr{C}(\mathbf{R}^k) & \xrightarrow{\sigma^*} & \mathscr{C}(\mathbf{R}^n, \sigma). \end{array}$$

It follows that the bottom arrow is split surjective, since the vertical equalities are identifications of topological vector spaces.

This completes the inductive step in the case $(\mathbf{R}^n)^\Gamma \neq 0$. From now on, we suppose $(\mathbf{R}^n)^\Gamma = 0$.

Let $F = \sigma^{-1}(0)$. The proof of (**) in (3.3) of [13] applies without change, and we get that

$$(\sigma|\mathbf{R}^n - F)^*: \mathscr{C}(\mathbf{R}^k - 0) \to \mathscr{C}(\mathbf{R}^n - F; \sigma) \tag{1}$$

is split surjective. The formula (**) of [13], (3.3) says that this homomorphism is surjective. Our stronger inductive hypothesis permits the stronger conclusion.

We let S^{k-1}, Σ^{n-1}, r_+, ρ_+ be defined exactly as in §8. In the case $F \neq \emptyset$, Σ^{n-1} is non-compact, but it still is an analytic manifold. We let $\mathbf{R}_{++}$ denote the positive real numbers. then

$$r_+: S^{k-1} \times \mathbf{R}_{++} \to \mathbf{R}^k - 0, \qquad \rho_+: \Sigma^{n-1} \times \mathbf{R}_{++} \to \mathbf{R}^n - F$$

are analytic diffeomorphisms. Hence from the fact that (1) is split surjective we get that

$$(\sigma \times \mathrm{id})^*: \mathscr{C}(S^{k-1} \times \mathbf{R}_{++}) \to \mathscr{C}(\Sigma^{n-1} \times \mathbf{R}_{++}; \sigma \times \mathrm{id})$$

is split surjective. It follows easily that

$$\sigma^*: \mathscr{C}(S^{k-1}) \to \mathscr{C}(\Sigma^{n-1}; \sigma)$$

is split surjective. Hence, it follows just as in §8 that the top arrow in the following diagram is split surjective:

$$\begin{tikzcd} \mathscr{C}(S^{k-1} \times R_+)_{S^{k-1}\times 0} \arrow[r, "(\sigma\times id)^*"] & \mathscr{C}(\Sigma^{n-1} \times R_+; \sigma \times id)_{\Sigma^{n-1}\times 0} \\ \mathscr{C}(\mathbf{R}^k)_0 \arrow[u, "r_+^*"'] \arrow[u, bend left] \arrow[r, "\sigma^*"] & \mathscr{C}(\mathbf{R}^n; \sigma)_0. \arrow[u, "\rho_+^*"'] \end{tikzcd}$$

It follows immediately that the bottom arrow is split surjective, since the right vertical arrow is clearly injective.

To complete the proof of Theorem 4, we consider the diagram of §9, with G replaced by Γ. We have shown that the left vertical arrow is split surjective and it is easily seen that the right vertical arrow is split surjective. The assertion we must prove is that the middle vertical arrow is split surjective. By the argument of §9, it is enough to show that the mapping $ST: \mathscr{C}(\mathbf{R}^n)^\Gamma \to F(\mathbf{R}^k)$, obtained by composing $T: \mathscr{C}(\mathbf{R}^n)^\Gamma \to F(\mathbf{R}^n)^\Gamma$ and the splitting mapping $S: F(\mathbf{R}^n)^\Gamma \to F(\mathbf{R}^k)$, lifts to a mapping $\mathscr{C}(\mathbf{R}^n)^\Gamma \to \mathscr{C}(\mathbf{R}^k)$.

But, $F(\mathbf{R}^n)^\Gamma \subset F(\mathbf{R}^n)$ is split injective, so S extends to a continuous linear mapping $\tilde{S}: F(\mathbf{R}^n) \to F(\mathbf{R}^k)$. Then $\tilde{S}T: \mathscr{C}(\mathbf{R}^n) \to F(\mathbf{R}^k)$ extends ST, and the lemma of §7 shows that this mapping lifts to a continuous linear mapping $\mathscr{C}(\mathbf{R}^n) \to \mathscr{C}(\mathbf{R}^k)$. The restriction of this lifting to $\mathscr{C}(\mathbf{R}^n)^\Gamma$ gives the desired lifting. □

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Department of Mathematics,
Princeton University